



# Isospectral Dirac operators

Yuri Ashrafyan and Tigran Harutyunyan 

Yerevan State University, Alex Manoogian 1, Yerevan, 0025, Armenia

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**Abstract.** We give the description of self-adjoint regular Dirac operators, on  $[0, \pi]$ , with the same spectra.

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## 1 Introduction and statement of result

Let  $p$  and  $q$  are real-valued, summable on  $[0, \pi]$  functions, i.e.  $p, q \in L^1_{\mathbb{R}}[0, \pi]$ . By  $L(p, q, \alpha) = L(\Omega, \alpha)$  we denote the boundary-value problem for canonical Dirac system (see [5, 6, 9, 13, 14]):

$$\ell y \equiv \left\{ B \frac{d}{dx} + \Omega(x) \right\} y = \lambda y, \quad x \in (0, \pi), \quad y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}, \quad \lambda \in \mathbb{C}, \quad (1.1)$$

$$y_1(0) \cos \alpha + y_2(0) \sin \alpha = 0, \quad \alpha \in \left( -\frac{\pi}{2}, \frac{\pi}{2} \right], \quad (1.2)$$

$$y_1(\pi) = 0, \quad (1.3)$$


where

$$B = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \Omega(x) = \begin{pmatrix} p(x) & q(x) \\ q(x) & -p(x) \end{pmatrix}.$$

By the same  $L(p, q, \alpha)$  we also denote a self-adjoint operator generated by differential expression  $\ell$  in Hilbert space of two component vector-function  $L^2([0, \pi]; \mathbb{C}^2)$  on the domain

$$D = \left\{ y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}; y_k \in AC[0, \pi], (\ell y)_k \in L^2[0, \pi], k = 1, 2; \right. \\ \left. y_1(0) \cos \alpha + y_2(0) \sin \alpha = 0, \quad y_1(\pi) = 0 \right\}$$

where  $AC[0, \pi]$  is the set of absolutely continuous functions on  $[0, \pi]$  (see, e.g. [13, 16]). It is well known (see [1, 5, 9]) that under these conditions the spectra of the operator  $L(p, q, \alpha)$  is purely discrete and consists of simple, real eigenvalues, which we denote by  $\lambda_n = \lambda_n(p, q, \alpha) =$

 Corresponding author. Email: hartigr@yahoo.co.uk

$\lambda_n(\Omega, \alpha)$ ,  $n \in \mathbb{Z}$ , to emphasize the dependence of  $\lambda_n$  on quantities  $p, q$  and  $\alpha$ . It is also well known (see, e.g. [1, 5, 9]) that the eigenvalues form a sequence, unbounded below as well as above. So we will enumerate it as  $\lambda_k < \lambda_{k+1}$ ,  $k \in \mathbb{Z}$ ,  $\lambda_k > 0$ , when  $k > 0$  and  $\lambda_k < 0$ , when  $k < 0$ , and the nearest to zero eigenvalue we will denote by  $\lambda_0$ . If there are two nearest to zero eigenvalue, then by  $\lambda_0$  we will denote the negative one. With this enumeration it is proved (see [1, 5, 9]), that the eigenvalues have the asymptotics:

$$\lambda_n(\Omega, \alpha) = n - \frac{\alpha}{\pi} + r_n, \quad r_n = o(1), \quad n \rightarrow \pm\infty. \quad (1.4)$$

In what follows, writing  $\Omega \in A$  will mean  $p, q \in A$ . If  $\Omega \in L^2_{\mathbb{R}}[0, \pi]$ , then we know, (see, e.g. [9]), that instead of  $r_n = o(1)$  we have:

$$\sum_{n=-\infty}^{\infty} r_n^2 < \infty. \quad (1.5)$$

Let  $\varphi(x, \lambda) = \varphi(x, \lambda, \alpha, \Omega)$  be the solution of the Cauchy problem

$$\ell\varphi = \lambda\varphi, \quad \varphi(0, \lambda) = \begin{pmatrix} \sin \alpha \\ -\cos \alpha \end{pmatrix}. \quad (1.6)$$

Since the differential expression  $\ell$  self-adjoint, then the components  $\varphi_1(x, \lambda)$  and  $\varphi_2(x, \lambda)$  of the vector-function  $\varphi(x, \lambda)$  we can choose real-valued for real  $\lambda$ . By  $a_n = a_n(\Omega, \alpha)$  we denote the squares of the  $L^2$ -norm of the eigenfunctions  $\varphi_n(x, \Omega) = \varphi(x, \lambda_n(\Omega, \alpha), \alpha, \Omega)$ :

$$a_n = \|\varphi_n\|^2 = \int_0^\pi |\varphi_n(x, \Omega)|^2 dx, \quad n \in \mathbb{Z}.$$

The numbers  $a_n$  are called norming constants. And by  $h_n(x, \Omega)$  we will denote normalized eigenfunctions (i.e.  $\|h_n(x)\| = 1$ ):

$$h_n(x, \Omega) = h_n(x) = \frac{\varphi_n(x, \Omega)}{\sqrt{a_n(\Omega, \alpha)}}. \quad (1.7)$$

It is known (see [5, 9]) that in the case of  $\Omega \in L^2_{\mathbb{R}}[0, \pi]$  the norming constants have an asymptotic form:

$$a_n(\Omega) = \pi + c_n, \quad \sum_{n=-\infty}^{\infty} c_n^2 < \infty. \quad (1.8)$$

**Definition 1.1.** Two Dirac operators  $L(\Omega, \alpha)$  and  $L(\tilde{\Omega}, \tilde{\alpha})$  are said to be isospectral, if  $\lambda_n(\Omega, \alpha) = \lambda_n(\tilde{\Omega}, \tilde{\alpha})$ , for every  $n \in \mathbb{Z}$ .

**Lemma 1.2.** Let  $\Omega, \tilde{\Omega} \in L^1_{\mathbb{R}}[0, \pi]$  and the operators  $L(\Omega, \alpha)$  and  $L(\tilde{\Omega}, \tilde{\alpha})$  are isospectral. Then  $\tilde{\alpha} = \alpha$ .

*Proof.* The proof follows from the asymptotics (1.4):

$$\frac{\alpha}{\pi} = \lim_{n \rightarrow \infty} (n - \lambda_n(\Omega, \alpha)) = \lim_{n \rightarrow \infty} (n - \lambda_n(\tilde{\Omega}, \tilde{\alpha})) = \frac{\tilde{\alpha}}{\pi}.$$

□

So, instead of isospectral operators  $L(\Omega, \alpha)$  and  $L(\tilde{\Omega}, \tilde{\alpha})$ , we can talk about “isospectral potentials”  $\Omega$  and  $\tilde{\Omega}$ .

**Theorem 1.3** (Uniqueness theorem). *The map*

$$(\Omega, \alpha) \in L_{\mathbb{R}}^2[0, \pi] \times \left(-\frac{\pi}{2}, \frac{\pi}{2}\right] \longleftrightarrow \{\lambda_n(\Omega, \alpha), a_n(\Omega, \alpha); n \in \mathbb{Z}\}$$

*is one-to-one.*

**Remark 1.4.** It is natural to call this a Marchenko theorem, since it is an analogue of the famous theorem of V. A. Marchenko [15], in the case for Sturm–Liouville problem. The proof of this theorem for the case  $p, q \in AC[0, \pi]$  there is in the paper [18]. The detailed proof for the case  $p, q \in L_{\mathbb{R}}^2[0, \pi]$  there is in [7] (see also [4–6, 8, 10, 19]).

Let us fix some  $\Omega \in L_{\mathbb{R}}^2[0, \pi]$  and consider the set of all canonical potentials  $\tilde{\Omega} = \begin{pmatrix} \tilde{p} & \tilde{q} \\ \tilde{q} & -\tilde{p} \end{pmatrix}$ , with the same spectra as  $\Omega$ :

$$M^2(\Omega) = \{\tilde{\Omega} \in L_{\mathbb{R}}^2[0, \pi] : \lambda_n(\tilde{\Omega}, \tilde{\alpha}) = \lambda_n(\Omega, \alpha), n \in \mathbb{Z}\}.$$

Our main goal is to give the description of the set  $M^2(\Omega)$  as explicit as it possible.

From the uniqueness theorem the next corollary easily follows.

**Corollary 1.5.** *The map*

$$\tilde{\Omega} \in M^2(\Omega) \leftrightarrow \{a_n(\tilde{\Omega}), n \in \mathbb{Z}\}$$

*is one-to-one.*

Since  $\tilde{\Omega} \in M^2(\Omega)$ , then  $a_n(\tilde{\Omega})$  have similar to (1.8) asymptotics. Since  $a_n(\Omega)$  and  $a_n(\tilde{\Omega})$  are positive numbers, there exist real numbers  $t_n = t_n(\tilde{\Omega})$ , such that  $\frac{a_n(\Omega)}{a_n(\tilde{\Omega})} = e^{t_n}$ . From the latter equality and from (1.8) follows that

$$e^{t_n} = 1 + d_n, \quad \sum_{n=-\infty}^{\infty} d_n^2 < \infty. \quad (1.9)$$

It is easy to see, that the sequence  $\{t_n; n \in \mathbb{Z}\}$  is also from  $l^2$ , i.e.  $\sum_{n=-\infty}^{\infty} t_n^2 < \infty$ . Since all  $a_n(\Omega)$  are fixed, then from the corollary 1.5 and the equality  $a_n(\tilde{\Omega}) = a_n(\Omega)e^{-t_n}$  we will get the following corollary.

**Corollary 1.6.** *The map*

$$\tilde{\Omega} \in M^2(\Omega) \leftrightarrow \{t_n(\tilde{\Omega}), n \in \mathbb{Z}\} \in l^2$$

*is one-to-one.*

Thus, each isospectral potential is uniquely determined by a sequence  $\{t_n; n \in \mathbb{Z}\}$ . Note, that the problem of description of isospectral Sturm–Liouville operators was solved in [3, 11, 12, 17].

For Dirac operators the description of  $M^2(\Omega)$  is given in [8]. This description has a “recurrent” form, i.e. at the first in [8] is given the description of a family of isospectral potentials  $\Omega(x, t)$ ,  $t \in \mathbb{R}$ , for which only one norming constant  $a_m(\Omega(\cdot, t))$  different from  $a_m(\Omega)$  (namely,  $a_m(\Omega(\cdot, t)) = a_m(\Omega)e^{-t}$ ), while the others are equal, i.e.  $a_m(\Omega(\cdot, t)) = a_m(\Omega)$ , when  $n \neq m$ .

**Theorem 1.7** ([8]). Let  $t \in \mathbb{R}$ ,  $\alpha \in (-\frac{\pi}{2}, \frac{\pi}{2}]$  and

$$\Omega(x, t) = \Omega(x) + \frac{e^t - 1}{\theta_m(x, t, \Omega)} \{Bh_m(x, \Omega)h_m^*(x, \Omega) - h_m(x, \Omega)h_m^*(x, \Omega)B\},$$

where  $\theta_m(x, t, \Omega) = 1 + (e^t - 1) \int_0^x |h_m(s, \Omega)|^2 ds$ , and  $*$  is a sign of transposition, e.g.  $h_m^* = \begin{pmatrix} h_{m_1} \\ h_{m_2} \end{pmatrix}^* = (h_{m_1}, h_{m_2})$ . Then, for arbitrary  $t \in \mathbb{R}$ ,  $\lambda_n(\Omega, t) = \lambda_n(\Omega)$  for all  $n \in \mathbb{Z}$ ,  $a_n(\Omega, t) = a_n(\Omega)$  for all  $n \in \mathbb{Z} \setminus \{m\}$  and  $a_m(\Omega, t) = a_m(\Omega)e^{-t}$ . The normalized eigenfunctions of the problem  $L(\Omega(\cdot, t), \alpha)$  are given by the formulae:

$$h_n(x, \Omega(\cdot, t)) = \begin{cases} \frac{e^{-t/2}}{\theta_m(x, t, \Omega)} h_m(x, \Omega), & \text{if } n = m, \\ h_n(x, \Omega) - \frac{(e^t - 1) \int_0^x h_m^*(s, \Omega) h_n(s, \Omega) ds}{\theta_m(x, t, \Omega)} h_m(x, \Omega), & \text{if } n \neq m. \end{cases}$$

Theorem 1.7 shows that it is possible to change exactly one norming constant, keeping the others. As examples of isospectral potentials  $\Omega$  and  $\tilde{\Omega}$  we can present  $\Omega(x) \equiv 0 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$  and

$$\tilde{\Omega}(x) = \Omega_{m,t}(x) = \frac{\pi(e^t - 1)}{\pi + (e^t - 1)x} \begin{pmatrix} -\sin 2mx & \cos 2mx \\ \cos 2mx & \sin 2mx \end{pmatrix},$$

where  $t \in \mathbb{R}$  is an arbitrary real number and  $m \in \mathbb{Z}$  is an arbitrary integer.

Changing successively each  $a_m(\Omega)$  by  $a_m(\Omega)e^{-t_m}$ , we can obtain any isospectral potential, corresponding to the sequence  $\{t_m; m \in \mathbb{Z}\} \in l^2$ . It follows from the uniqueness Theorem 1.3 that the sequence, in which we change the norming constants, is not important.

In [8] were used the following designations:

$$\begin{aligned} T_{-1} &= \{\dots, 0, \dots\}, \\ T_0 &= \{\dots, 0, \dots, 0, t_0, 0, \dots, 0, \dots\}, \\ T_1 &= \{\dots, 0, \dots, 0, 0, t_0, t_1, 0, \dots, 0, \dots\}, \\ T_2 &= \{\dots, 0, \dots, 0, t_{-1}, t_0, t_1, 0, \dots, 0, \dots\}, \\ &\vdots \\ T_{2n} &= \{\dots, 0, 0, t_{-n}, \dots, t_{-1}, t_0, t_1, \dots, t_{n-1}, t_n, 0, \dots\}, \\ T_{2n+1} &= \{\dots, 0, t_{-n}, t_{-n+1}, \dots, t_{-1}, t_0, t_1, \dots, t_n, t_{n+1}, 0, \dots\}, \\ &\vdots \end{aligned}$$

Let  $\Omega(x, T_{-1}) \equiv \Omega(x)$  and

$$\Omega(x, T_m) = \Omega(x, T_{m-1}) + \Delta\Omega(x, T_m), \quad m = 0, 1, 2, \dots,$$

where

$$\Delta\Omega(x, T_m) = \frac{e^{t_{\tilde{m}}} - 1}{\theta_m(x, t_{\tilde{m}}, \Omega(\cdot, T_{m-1}))} [Bh_{\tilde{m}}(x, \Omega(\cdot, T_{m-1}))h_{\tilde{m}}^*(\cdot) - h_{\tilde{m}}(\cdot)h_{\tilde{m}}^*(\cdot)B],$$

where  $\tilde{m} = \frac{m+1}{2}$ , if  $m$  is odd and  $\tilde{m} = -\frac{m}{2}$ , if  $m$  is even. The arguments in others  $h_{\tilde{m}}(\cdot)$  and  $h_{\tilde{m}}^*(\cdot)$  are the same as in the first. And after that in [8] the following theorem was proved.

**Theorem 1.8** ([8]). Let  $T = \{t_n, n \in \mathbb{Z}\} \in l^2$  and  $\Omega \in L^2_{\mathbb{R}}[0, \pi]$ . Then

$$\Omega(x, T) \equiv \Omega(x) + \sum_{m=0}^{\infty} \Delta\Omega(x, T_m) \in M^2(\Omega). \quad (1.10)$$

We see, that each potential matrix  $\Delta\Omega(x, T_m)$  defined by normalized eigenfunctions  $h_{\tilde{m}}(x, \Omega(x, T_{m-1}))$  of the previous operator  $L(\Omega(\cdot, T_{m-1}), \alpha)$ . This approach we call “recurrent” description.

In this paper, we want to give a description of the set  $M^2(\Omega)$  only in terms of eigenfunctions  $h_n(x, \Omega)$  of the initial operator  $L(\Omega, \alpha)$  and sequence  $T \in l^2$ . With this aim, let us denote by  $N(T_m)$  the set of the positions of the numbers in  $T_m$ , which are not necessary zero, i.e.

$$\begin{aligned} N(T_0) &= \{0\}, \\ N(T_1) &= \{0, 1\}, \\ N(T_2) &= \{-1, 0, 1\}, \\ &\vdots \\ N(T_{2n}) &= \{-n, -(n-1), \dots, 0, \dots, n-1, n\}, \\ N(T_{2n+1}) &= \{-n, -(n-1), \dots, 0, \dots, n, n+1\}, \\ &\vdots \end{aligned}$$

in particular  $N(T) \equiv \mathbb{Z}$ . By  $S(x, T_m)$  we denote the  $(m+1) \times (m+1)$  square matrix

$$S(x, T_m) = \left( \delta_{ij} + (e^{t_j} - 1) \int_0^x h_i^*(s) h_j(s) ds \right)_{i,j \in N(T_m)} \quad (1.11)$$

where  $\delta_{ij}$  is a Kronecker symbol. By  $S_p^{(k)}(x, T_m)$  we denote a matrix which is obtained from the matrix  $S(x, T_m)$  by replacing the  $k$ th column of  $S(x, T_m)$  by  $H_p(x, T_m) = \{-(e^{t_k} - 1)h_{k_p}(x)\}_{k \in N(T_m)}$  column,  $p = 1, 2$ . Now we can formulate our result as follows.

**Theorem 1.9.** Let  $T = \{t_k\}_{k \in \mathbb{Z}} \in l^2$  and  $\Omega \in L^2_{\mathbb{R}}[0, \pi]$ . Then the isospectral potential from  $M^2(\Omega)$ , corresponding to  $T$ , is given by the formula

$$\Omega(x, T) = \Omega(x) + G(x, x, T)B - BG(x, x, T) = \begin{pmatrix} p(x, T) & q(x, T) \\ q(x, T) & -p(x, T) \end{pmatrix}, \quad (1.12)$$

where

$$G(x, x, T) = \frac{1}{\det S(x, T)} \sum_{k \in \mathbb{Z}} \left( \frac{\det S_1^{(k)}(x, T)}{\det S_2^{(k)}(x, T)} \right) h_k^*(x),$$

and  $\det S(x, T) = \lim_{m \rightarrow \infty} \det S(x, T_m)$  (the same for  $\det S_p^{(k)}(x, T)$ ,  $p = 1, 2$ ).

In addition, for  $p(x, T)$  and  $q(x, T)$  we get explicit representations:

$$\begin{aligned} p(x, T) &= p(x) - \frac{1}{\det S(x, T)} \sum_{k \in \mathbb{Z}} \sum_{p=1}^2 \det S_p^{(k)}(x, T) h_{k_{(3-p)}}(x), \\ q(x, T) &= q(x) + \frac{1}{\det S(x, T)} \sum_{k \in \mathbb{Z}} \sum_{p=1}^2 (-1)^{p-1} S_p^{(k)}(x, T) h_{k_p}(x). \end{aligned}$$

## 2 Proof of Theorem 1.9

The spectral function of an operator  $L(\Omega, \alpha)$  defined as

$$\rho(\lambda) = \begin{cases} \sum_{0 < \lambda_n \leq \lambda} \frac{1}{a_n(\Omega)}, & \lambda > 0, \\ -\sum_{\lambda < \lambda_n \leq 0} \frac{1}{a_n(\Omega)}, & \lambda < 0, \end{cases}$$

i.e.  $\rho(\lambda)$  is left-continuous, step function with jumps in points  $\lambda = \lambda_n$  equals  $\frac{1}{a_n}$  and  $\rho(0) = 0$ .

Let  $\Omega, \tilde{\Omega} \in L^2_{\mathbb{R}}[0, \pi]$  and they are isospectral. It is known (see [1, 2, 6, 13]), that there exists a function  $G(x, y)$  such that:

$$\varphi(x, \lambda, \alpha, \tilde{\Omega}) = \varphi(x, \lambda, \alpha, \Omega) + \int_0^x G(x, s) \varphi(s, \lambda, \alpha, \Omega) ds. \quad (2.1)$$

It is also known (see, e.g. [1, 6, 13]), that the function  $G(x, y)$  satisfies to the Gelfand–Levitan integral equation:

$$G(x, y) + F(x, y) + \int_0^x G(x, s) F(s, y) ds = 0, \quad 0 \leq y \leq x, \quad (2.2)$$

where

$$F(x, y) = \int_{-\infty}^{\infty} \varphi(x, \lambda, \alpha, \Omega) \varphi^*(y, \lambda, \alpha, \Omega) d[\tilde{\rho}(\lambda) - \rho(\lambda)]. \quad (2.3)$$

If the potential  $\tilde{\Omega}$  from  $M^2(\Omega)$  is such that only finite norming constants of the operator  $L(\tilde{\Omega}, \alpha)$  are different from the norming constants of the operator  $L(\Omega, \alpha)$ , i.e.  $a_n(\tilde{\Omega}) = a_n(\Omega) e^{-t_n}$ ,  $n \in N(T_m)$  and the others are equal, then it means, that

$$d\tilde{\rho}(\lambda) - d\rho(\lambda) = \sum_{k \in N(T_m)} \left( \frac{1}{\tilde{a}_k} - \frac{1}{a_k} \right) \delta(\lambda - \lambda_k) d\lambda = \sum_{k \in N(T_m)} \left( \frac{e^{t_k} - 1}{a_k} \right) \delta(\lambda - \lambda_k) d\lambda, \quad (2.4)$$

where  $\delta$  is Dirac  $\delta$ -function. In this case the kernel  $F(x, y)$  can be written in a form of a finite sum (using notation (1.7)):

$$F(x, y) = F(x, y, T_m) = \sum_{k \in N(T_m)} (e^{t_k} - 1) h_k(x, \Omega) h_k^*(y, \Omega), \quad (2.5)$$

and consequently, the integral equation (2.2) becomes to an integral equation with degenerated kernel, i.e. it becomes to a system of linear equations and we will look for the solution in the following form:

$$G(x, y, T_m) = \sum_{k \in N(T_m)} g_k(x) h_k^*(y), \quad (2.6)$$

where  $g_k(x) = \begin{pmatrix} g_{k_1}(x) \\ g_{k_2}(x) \end{pmatrix}$  is an unknown vector-function. Putting the expressions (2.5) and (2.6) into the integral equation (2.2) we will obtain a system of algebraic equations for determining the functions  $g_k(x)$ :

$$g_k(x) + \sum_{i \in N(T_m)} s_{ik}(x) g_i(x) = -(e^{t_k} - 1) h_k(x), \quad k \in N(T_m), \quad (2.7)$$

where

$$s_{ik}(x) = (e^{t_k} - 1) \int_0^x h_i^*(s) h_k(s) ds.$$

It would be better if we consider the equations (2.7) for the vectors  $g_k = \begin{pmatrix} g_{k_1} \\ g_{k_2} \end{pmatrix}$  by coordinates  $g_{k_1}$  and  $g_{k_2}$  to be a system of scalar linear equations:

$$g_{k_p}(x) + \sum_{i \in N(T_m)} s_{ik}(x) g_{i_p}(x) = -(e^{t_k} - 1) h_{k_p}(x), \quad k \in N(T_m), \quad p = 1, 2. \quad (2.8)$$

The systems (2.8) might be written in matrix form

$$S(x, T_m) g_p(x, T_m) = H_p(x, T_m), \quad p = 1, 2, \quad (2.9)$$

where the column vectors  $g_p(x, T_m) = \{g_{k_p}(x, T_m)\}_{k \in N(T_m)}$ ,  $p = 1, 2$ , and the solution can be found in the form (Cramer's rule):

$$g_{k_p}(x, T_m) = \frac{\det S_p^{(k)}(x, T_m)}{\det S(x, T_m)}, \quad k \in N(T_m), \quad p = 1, 2.$$

Thus we have obtained for  $g_k(x)$  the following representation:

$$g_k(x, T_m) = \frac{1}{\det S(x, T_m)} \begin{pmatrix} \det S_1^{(k)}(x, T_m) \\ \det S_2^{(k)}(x, T_m) \end{pmatrix} \quad (2.10)$$

and then by putting (2.10) into (2.6) we find the  $G(x, y, T_m)$  function. If the potential  $\Omega$  is from  $L^1_{\mathbb{R}}$ , then such is also the kernel  $G(x, x, T_m)$  (see [8]), and the relation between them gives as follows:

$$\Omega(x, T_m) = \Omega(x) + G(x, x, T_m)B - BG(x, x, T_m). \quad (2.11)$$

On the other hand we have

$$\Omega(x, T_m) = \Omega(x) + \sum_{k=0}^m \triangle \Omega(x, T_k). \quad (2.12)$$

So, using the Theorem 1.8 and the equality (2.12) we can pass to the limit in (2.11), when  $m \rightarrow \infty$ :

$$\Omega(x, T) = \Omega(x) + G(x, x, T)B - BG(x, x, T). \quad (2.13)$$

The potentials  $\Omega(x, T)$  in (1.10) and (2.13) have the same spectral data  $\{\lambda_n(T), a_n(T)\}_{n \in \mathbb{Z}}$ , and therefore they are the same and  $\Omega(\cdot, T)$  defined by (2.13) is also from  $M^2(\Omega)$ .

Using (2.6) and (2.10) we calculate the expression  $G(x, x, T_m)B - BG(x, x, T_m)$  and pass to the limit, obtaining for the  $p(x, T)$  and  $q(x, T)$  the representations:

$$p(x, T) = p(x) - \frac{1}{\det S(x, T)} \sum_{k \in N(T)} \sum_{p=1}^2 \det S_p^{(k)}(x, T) h_{k(3-p)}(x),$$

$$q(x, T) = q(x) + \frac{1}{\det S(x, T)} \sum_{k \in N(T)} \sum_{p=1}^2 (-1)^{p-1} S_p^{(k)}(x, T) h_{k_p}(x).$$

Theorem 1.9 is proved.

For example, when we change just one norming constant (e.g. for  $T_0$ ) we get two independent linear equations:

$$(1 + s_{00}(x))g_{0_1}(x) = -(e^{t_0} - 1)h_{0_1}(x),$$

$$(1 + s_{00}(x))g_{0_2}(x) = -(e^{t_0} - 1)h_{0_2}(x).$$

For the solutions we get:

$$g_{0_1}(x) = -\frac{(e^{t_0} - 1)h_{0_1}(x)}{1 + s_{00}(x)},$$

$$g_{0_2}(x) = -\frac{(e^{t_0} - 1)h_{0_2}(x)}{1 + s_{00}(x)},$$

and for the potentials  $p(x, T_0)$  and  $q(x, T_0)$ :

$$p(x, T_0) = p(x) + \frac{e^{t_0} - 1}{1 + s_{00}(x)} (2h_{0_1}(x)h_{0_2}(x)),$$

$$q(x, T_0) = q(x) + \frac{e^{t_0} - 1}{1 + s_{00}(x)} (h_{0_2}^2(x) - h_{0_1}^2(x)).$$

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